Boundary Element Method ~~Cubic Spline~~ 10/15/2017 ~~24/4/2017~~

Advantages of BEM:

① Only boundary points are needed to be discretized.

② Effective in computing derivatives

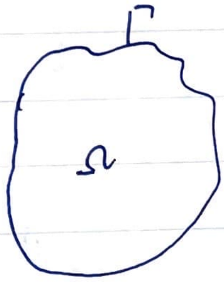
Disadvantage of BEM:

① Fundamental solution is needed. Non-linear problem is not suitable.

② Very dense matrix to be solved.

$$C_i u_i = -\sum_j \int_{\Gamma_j} u_j \frac{\partial v_i}{\partial n} d\Gamma + \sum_j \int_{\Gamma_j} \frac{\partial u_j}{\partial n} v_i \, d\Gamma$$

Gauss theorem



$$\int_\Omega \nabla \cdot u \, dV = \oint_\Gamma u \cdot n \, ds$$

let $u^y = 0$, $\int_\Omega \frac{\partial u}{\partial x} dV = \oint_\Gamma u n^x ds$, $\int_\Omega g \frac{\partial f}{\partial x} dV = -\int_\Omega f \frac{\partial g}{\partial x} dV + \int_\Gamma f g n^x ds$

let $u^x = 0$, $\int_\Omega \frac{\partial u}{\partial y} dV = \oint_\Gamma u n^y ds$, $\int_\Omega g \frac{\partial f}{\partial y} dV = -\int_\Omega f \frac{\partial g}{\partial x} dV + \int_\Gamma f g n^y ds$

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Green's second identity

$$\int_\Omega v\nabla^2 u \, dV = \int_\Omega [\nabla\cdot(v\nabla u) - \nabla v\cdot\nabla u] \, dV$$

$$= \oint_\Gamma v\nabla u\cdot n \, ds - \int_\Omega \nabla v\cdot\nabla u \, dV$$

similarly.

$$\int_\Omega u\nabla^2 v \, dV = \oint_\Gamma u\nabla v\cdot n \, ds - \int_\Omega \nabla u\cdot\nabla v \, dV$$

$$\therefore \int_\Omega (v\nabla^2 u - u\nabla^2 v) \, dV = \oint_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds$$

$$\frac{\partial}{\partial n} = n^x\frac{\partial}{\partial x} + n^y\frac{\partial}{\partial y}$$

Linear and adjoint operator

$$L(u) = \sum_{|k|\le p} a_k(x) D^k(u) \qquad k = (k_1, k_2, \ldots, k_n)$$
$$x = (x_1, \ldots, x_n)$$

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}$$

$$L(u+v) = L(u) + L(v)$$

~~or L(u) can be denoted by~~ $\cancel{\sum_{|k|\le p} D^k(a_k u)}$

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adjoint $L^*(u) = \sum_{|k|\le p} (-1)^{|k|} D^k(a_k u)$

let $L(u) = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu$

$$L^*(u) = \frac{\partial^2}{\partial x^2}(Au) + 2\frac{\partial^2}{\partial x\partial y}(Bu) + \frac{\partial^2}{\partial y^2}(Cu) - \frac{\partial}{\partial x}(Du) - \frac{\partial}{\partial y}(Eu) + Fu$$

$$\therefore \int_\Omega [vL(u) - uL^*(v)]\,dV$$

$$= \int_\Omega \Big[Av\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2}{\partial x^2}(Av) + 2Bv\frac{\partial^2 u}{\partial x\partial y} - 2u\frac{\partial^2}{\partial x\partial y}(Bv) + Cv\frac{\partial^2 u}{\partial y^2} - u\frac{\partial^2}{\partial y^2}(Cv)$$
$$+ Dv\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}(Dv) + Ev\frac{\partial u}{\partial y} + u\frac{\partial}{\partial y}(Ev)\Big]\,dV$$

$$= \oint_\Gamma \Big[Av\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}(Av) + 2Bv\frac{\partial u}{\partial y} + Duv\Big]n^x\,ds$$
$$+ \oint_\Gamma \Big[-2u\frac{\partial}{\partial x}(Bv) + Cv\frac{\partial u}{\partial y} - u\frac{\partial}{\partial y}(Cv) + Euv\Big]n^y\,ds$$

$Bv\frac{\partial^2 u}{\partial x\partial y}$

$Bv\frac{\partial u}{\partial y}n^x$, $Bv\frac{\partial u}{\partial x}n^y$

$u\frac{\partial^2}{\partial x\partial y}(Bv)$

$u\frac{\partial}{\partial x}(Bv)n^y$, $u\frac{\partial}{\partial y}(Bv)n^x$

$$= \oint_\Gamma \Big[Av\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}(Av) + Bv\Big(\frac{\partial u}{\partial y}\Big) - u\frac{\partial}{\partial y}(Bv) + Duv\Big]n^x\,ds$$
$$+ \oint_\Gamma \Big[Bv\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}(Bv) + Cv\frac{\partial u}{\partial y} - u\frac{\partial}{\partial y}(Cv) + Euv\Big]n^y\,ds$$

$$= \oint_\Gamma (Xn^x + Yn^y)\,ds$$

[General form of Green's second identity]

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$$X = A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv$$

$$Y = B\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + C\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)uv$$

if $L^* = L$, self-adjoint

Ex., $D = E = 0$, A, B, C = const

Poisson equation

$\nabla^2 u = f$ $\quad \Omega$

$u = \bar{u}$ $\quad \Gamma$ $\quad$ Dirichlet

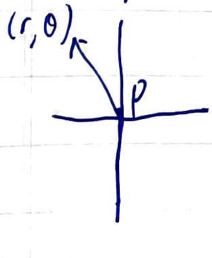
$\frac{\partial u}{\partial n} = \bar{u}_n$ $\quad \Gamma$ $\quad$ Neumann

Fundamental solution:

$$\nabla^2 u = \delta(\vec{x} - \vec{p}), \quad u = u(\vec{x})$$

2D:

Transformation to cylindrical coordinate, and let the singular point locate at the origin.



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \delta(\vec{x} - \vec{p})$$

$$\Downarrow$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0 \quad \text{at } R^2 \setminus (0,0)$$

symmetric with θ

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$$\therefore \frac{du}{dr} = \frac{A}{r}$$

$$u = A\ln r + B$$

Since $\nabla^2 u = \delta$, $\int_\Omega \nabla^2 u \, dV = \int_\Omega \delta \, dV = 1$

Gauss theorem $\quad \int_\Omega \nabla^2 u \, dV = \oint_{2\pi r} \nabla u \cdot ds = 2\pi r \cdot \frac{A}{r} = 2\pi A = 1$

let $r \to \infty$

$$\therefore A = \frac{1}{2\pi}$$

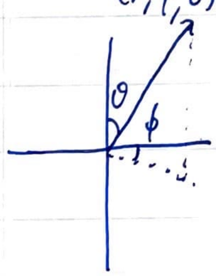
B is set to zero.

$$\therefore u = \frac{1}{2\pi}\ln r$$

3D:

Transformation to spherical coordinate, and let the singular point locate at the origin

(r, ϕ, θ) $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \delta(\vec{x} - \vec{P})$



symmetric in ϕ, θ,

$$\therefore \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = 0 \qquad R^2 \setminus (0,0)$$

$$r^2 \frac{du}{dr} = A \quad \to \quad u = -\frac{A}{r} + B$$

Since $\nabla^2 u = \delta$, $\int_\Omega \nabla^2 u \, dV = \int_\Omega \delta \, dV = 1$

Gauss theorem $\quad \int_\Omega \nabla^2 u \, dV = \lim_{r \to \infty} \int_{4\pi r^2} \frac{1}{r^2}\frac{d}{dr}(r^2 u) \, ds = \lim_{r\to\infty} 4\pi r^2 \cdot \frac{1}{r^2}\frac{d}{dr}(-rA)$

$$= -4\pi A = 1$$

$$\therefore A = -\frac{1}{4\pi}$$

$$B = 0$$

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$$\therefore \quad u = -\frac{1}{4\pi r}$$

BEM for Laplace equation (direct)

$$\nabla^2 u = 0$$
$$\nabla^2 v = \delta(\vec{x} - \vec{p})$$

apply Green's identity,

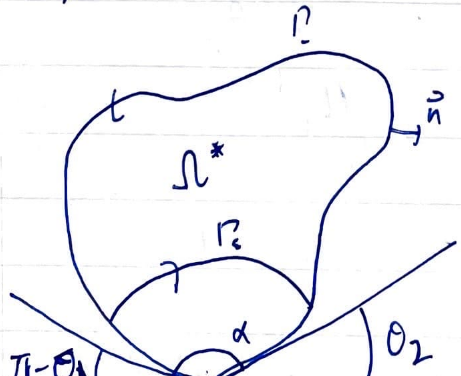
$$\int_\Omega (v \nabla^2 u - u \nabla^2 v) \, dV = \oint_\Gamma \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$

$$\therefore \quad u(\vec{p}) = -\oint_\Gamma \left(v(\vec{p}, \vec{q}) \frac{\partial u(\vec{q})}{\partial n} - u(\vec{q}) \frac{\partial v(\vec{p}, \vec{q})}{\partial n} \right) ds_q$$

$$\forall \vec{p} \in \Omega \setminus \Gamma$$

in order to find the relation between u and $\frac{\partial u}{\partial n}$ on Γ, move the $\vec{p}$ to boundary

2D



$$\Gamma = \Gamma_- \cup \Gamma_\varepsilon$$

$$\lim_{\varepsilon \to 0} (\theta_1 - \theta_2) = \alpha$$

$$\lim_{\varepsilon \to 0} \Gamma_\varepsilon = 0$$

$$|\Gamma_\varepsilon| = l, \quad \lim_{\varepsilon \to 0} l = 0$$

$$\lim_{\varepsilon \to 0} \Omega^* = \Omega$$

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apply the ~~g~~ Green's identity again,

$$\int_{\Omega^*} (v\nabla^2 u - u\nabla^2 v)\, dV = \int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_{\Gamma_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds$$

since $\vec{P}$ is located at the boundary, so LHS is always zero.

$$\therefore \lim_{\varepsilon\to 0}\left[\int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_{\Gamma_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds\right] = 0$$

Cauchy Principle Value

$$0 = \lim_{\varepsilon\to 0}\int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds$$

$$= \int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \lim_{\varepsilon\to 0}\left[\frac{r(\theta_1-\theta_2)}{2\pi}\frac{\partial u}{\partial n}\ln r + \frac{u\cos\phi}{2\pi r} r(\theta_2-\theta_1)\right]$$

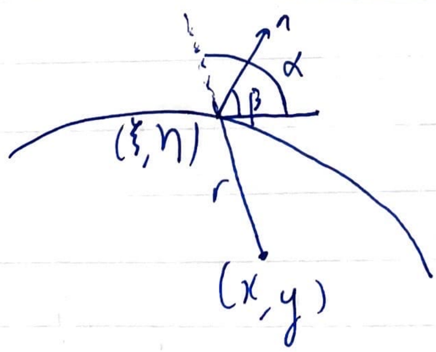
$\cos\phi = -1$

$\phi = \pi$

$\because \vec{n}$ of Γ_ε is opposite to $\vec{n}$ of integral

$$= \int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \frac{u\alpha}{2\pi} \qquad (ds = -r\,d\theta)$$

$$\therefore \frac{\alpha}{2\pi}u = -\int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds$$



$\phi = \beta - \alpha$

$n^x = \cos\beta$

$n^y = \sin\beta$

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$$

$$\frac{\partial r}{\partial x} = \frac{-(\xi - x)}{r} = -\cos\alpha \qquad \frac{\partial r}{\partial \xi} = \frac{\xi - x}{r} = \cos\alpha$$

$$\frac{\partial r}{\partial y} = -\frac{(\eta - y)}{r} = -\sin\alpha \qquad \frac{\partial r}{\partial \eta} = \frac{\eta - y}{r} = \sin\alpha$$

$$\therefore \frac{\partial r}{\partial n} = \frac{\partial r}{\partial \xi}n^x + \frac{\partial r}{\partial \eta}n^y = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \cos(\beta - \alpha) = \cos\phi$$

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$n^x \to +n^y$
$n^y \to -n^x$
$\circlearrowleft 90°$

$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial \xi}(-n^y) + \frac{\partial r}{\partial \eta} n^x$$
$$= -\cos\alpha \sin\beta + \sin\alpha \cos\beta$$
$$= -\sin(\beta - \alpha)$$
$$= -\sin\phi$$

$$\therefore \frac{\partial v}{\partial n} = \frac{1}{2\pi}\frac{1}{r}\frac{\partial r}{\partial n} = \frac{\cos\phi}{2\pi r} \qquad v = \frac{1}{2\pi}\ln r$$
$$\frac{\partial v}{\partial n} = -\frac{1}{4\pi}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial n} = \frac{\cos\phi}{4\pi r^2} \qquad v = \frac{1}{-4\pi r}$$

BEM for Poisson equation (direct)

$$\nabla^2 u = f$$

$$\therefore \varepsilon(P) u = \int_\Omega v f \, dV - \oint_\Gamma \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$

$$\varepsilon(P) = \begin{cases} 1 & P \in \Omega \setminus \Gamma \\ \frac{\alpha}{2\pi} & P \in \Gamma \\ 0 & P \in \mathbb{R}^2 \setminus \Omega \end{cases}$$

Discretization techniques (direct)

constant elements

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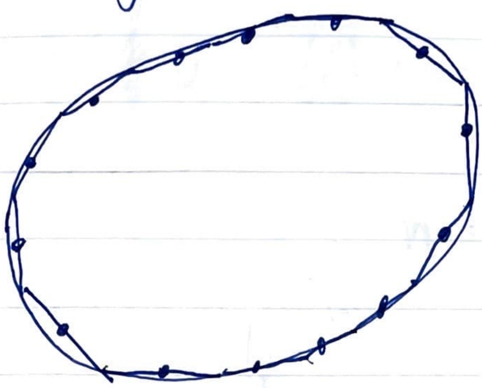

- nodes on center of line segment
- constant value on each line segment.

$\therefore \alpha = \pi$,

$$\frac{1}{2}u(\vec{p}) = -\oint_{\Gamma}\left(v(\vec{p},\vec{q})\frac{\partial u(\vec{q})}{\partial n} - u(\vec{q})\frac{\partial}{\partial n}\left(v(\vec{p},\vec{q})\right)\right)ds_q$$

$$\frac{1}{2}u(p_i) = -\sum_j \oint_{\Gamma_j}\left[v(p_i,q_j)\frac{\partial u(q_j)}{\partial n} - u(q_j)\frac{\partial}{\partial n}\left(v(p_i,q_j)\right)\right]ds_q$$

$u(p_i) = u_i$, $u(q_j) = u_j$ $\quad v(p_i,q_j) = v_{ij}$

$$\therefore \frac{1}{2}u_i = -\sum_j \oint_{\Gamma_j}\left[v_{ij}\frac{\partial u_j}{\partial n} - u_j\frac{\partial v_{ij}}{\partial n}\right]ds_j$$

$\oint \to \int$ wrong notation

$$= -\sum_j \frac{\partial u_j}{\partial n}\oint v_{ij}\,ds_j + \sum_j u_j \oint \frac{\partial v_{ij}}{\partial n}ds_j$$

let $\hat{H}_{ij} = \int_{\Gamma_j}\frac{\partial v_{ij}}{\partial n}ds_j \qquad G_{ij} = \int_{\Gamma_j} v_{ij}\,ds_j$

$$\hat{H}_{ii} = \int_{\Gamma_i}\frac{\partial v_{ii}}{\partial n}ds_i = \int_{\Gamma_i}\frac{\partial v_{ii}}{\partial r}\frac{\partial r}{\partial n}ds_i = 0$$

$$G_{ii} = \int_{\Gamma_i} v_{ii}\,ds_i = 2\int_0^{\frac{l}{2}}\frac{1}{2\pi}\ln r\,dr$$

$$= \frac{2}{2\pi}\left\{\left[r\ln r\right]_0^{\frac{l}{2}} - \int_0^{\frac{l}{2}}dr\right\} = \frac{l}{2\pi}\left(\ln\frac{l}{2} - 1\right)$$

Boundary Element Method (LU) 14/5/2017

LU decomposition

$$A = (a_{ij})_{i=1,N,\ j=1,N}$$

let $l_{in} = \dfrac{a_{i,n}}{a_{n,n}}$, $A^{(0)} = A$

$$\therefore L_n = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ 0 & & -l_{n+1,n} & \ddots & \\ & & \vdots & 0 & 1 \\ & & -l_{N,n} & & 1 \end{pmatrix}$$

$L_n A^{(n-1)} = A^{(n)}$, Ex. $\begin{pmatrix} 1 & & 0 \\ -l_{2,1} & \ddots & \\ \vdots & & \\ -l_{N,1} & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots \\ a_{21} & \\ \vdots & \\ a_{N1} & \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}$

$$\therefore U = L_{N-1} L_{N-2} \cdots L_2 L_1 A$$

$$\therefore A = L_1^{-1} L_2^{-1} \cdots L_{N-2}^{-1} L_{N-1}^{-1} U$$
$$= LU$$

$$L_n^{-1} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ 0 & & l_{n+1,n} & \ddots & \\ & & \vdots & & \\ & & l_{N,n} & & 1 \end{pmatrix}$$

$$L_n^{-1} L_n = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & l_{n+1,n} & \ddots \\ & & l_{N,n} & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & -l_{n+1,n} & \ddots \\ & & -l_{N,n} & & 1 \end{pmatrix}$$
$$= I_N$$

$$L = \begin{pmatrix} 1 & & & 0 \\ l_{2,1} & 1 & & \\ \vdots & l_{3,2} & \ddots & \\ \vdots & \vdots & & 1 \\ l_{N,1} & l_{N,2} & \cdots & 1 \end{pmatrix}$$

$Ax = b$

$LUx = b$

$Ux = L^{-1}b$

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$$\therefore \sum_j \left(\cancel{} \hat{H}_{ij} - \tfrac{1}{2}\delta_{ij} \right) u_j = \sum_j G_{ij} \frac{\partial u_j}{\partial n}$$

$$\cancel{\int_{\Gamma_j} \frac{\partial v_{ij}}{\partial n} ds_j = \int_{\Gamma_j} \frac{1}{2\pi} \frac{\cos\phi}{r} ds_j = \int_a^b \frac{1}{2\pi} \frac{\cos\phi}{r} dr}$$

$$\cancel{\text{let } r' = \frac{2}{b-a}\left(r - \frac{a+b}{2}\right)} \qquad \cancel{r = \frac{1}{2}\left((b-a)r' + a + b\right)}$$

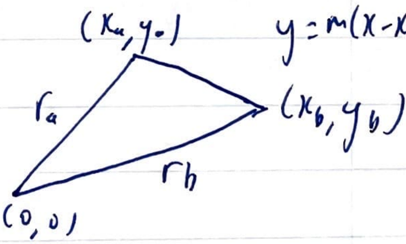
$$\cancel{dr' = \frac{2}{b-a} dr}$$

$$\cancel{\therefore \int_{\Gamma_j} \frac{\partial v_{ij}}{\partial n} ds_j = \frac{b-a}{2\pi} \int_{-1}^{1} \frac{\cos\phi}{(b-a)r + a + b} dr}$$

$$\cancel{\int_{\Gamma_j} v_{ij}\, ds_j = \int_a^b \frac{1}{2\pi} \ln r\, dr = \frac{b-a}{4\pi} \int_{-1}^{1} \{\ln[(b-a)r + a + b] - \ln 2\} dr}$$

$$\int_{\Gamma_j} v_{ij}\, ds_j \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad \tan\theta = \frac{dy}{dx} = m, \quad r = \sqrt{x^2 + y^2}$$

$$= \int_{x_a}^{x_b} \frac{\ln r}{2\pi} \sqrt{1+m^2}\, |dx|, \quad |\theta| \neq \frac{\pi}{2}$$



$$y = m(x - x_a) + y_a$$

$$= \frac{\sqrt{1+m^2}}{4\pi} \int_{x_a}^{x_b} \ln r^2 |dx|$$

$$= \frac{(x_b - x_a)\sqrt{1+m^2}}{8\pi} \int_{-1}^{1} I\, |dx'|$$

$$\text{let } x' = \frac{2}{x_b - x_a}\left(x - \frac{x_a + x_b}{2}\right)$$

$$dx = \frac{x_b - x_a}{2} dx'$$

$$I = \ln[x^2 + y^2]$$

$$x = \frac{1}{2}\left[(x_b - x_a)x' + x_a + x_b\right]$$

if $|m| = \infty$, $x_a = x_b$

$$\int_{\Gamma_j} v_{ij}\, ds_j = \frac{\cancel{}}{4\pi} \int_{y_a}^{y_b} \ln r^2 |dy| = \frac{(y_b - y_a)\cancel{}}{8\pi} \int_{-1}^{1} \ln[x^2 + y^2] |dy'|$$

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analytic solution:

$y = mx + c$

$m = \frac{y_b - y_a}{x_b - x_a}$

$c = -mx_a + y_a$

$$\int_{\Gamma_j} \frac{\ln r}{2\pi} ds_j = \frac{\sqrt{1+m^2}}{4\pi}\int_{x_a}^{x_b} \ln(x^2+y^2)\, dx$$

$$= \frac{\sqrt{1+m^2}}{4\pi}\int_{x_a}^{x_b} \ln(x^2+m^2x^2+2mxc+c^2)\, dx$$

$$= \frac{\sqrt{1+m^2}}{4\pi}\int_{x_a}^{x_b} \left[\ln(1+m^2)+\ln\left(x^2+\frac{2mc}{1+m^2}x+\frac{c^2}{1+m^2}\right)\right] dx$$

$$= \frac{(x_b-x_a)\sqrt{1+m^2}}{4\pi}\ln(1+m^2) + \frac{\sqrt{1+m^2}}{4\pi}\int_{x_a}^{x_b} \ln\left[\left(x+\frac{mc}{1+m^2}\right)^2+\frac{c^2}{1+m^2} - \frac{m^2c^2}{(1+m^2)^2}\right] dx$$

$$= A + \frac{\sqrt{1+m^2}}{4\pi}\int_{x_a}^{x_b} \ln\left[\left(x-\frac{mc}{1+m^2}\right)^2+\frac{1}{(1+m^2)^2}\right] dx$$

$x = x - \frac{mc}{1+m^2}$

let $g = \frac{mc}{1+m^2}$

$G^2 = \frac{1}{(1+m^2)^2}$

$G \geq 0$

$$= A + \frac{\sqrt{1+m^2}}{4\pi}\int_{x_a-g}^{x_b-g} \ln(x^2+G^2)\, dx$$

let $x = G\tan\theta$

$dx = G\sec^2\theta\, d\theta$

$$= A + \frac{\sqrt{1+m^2}}{4\pi}\int_{\theta_a}^{\theta_b} [2\ln G + 2\ln(\sec\theta)]\, G\sec^2\theta\, d\theta$$

$$= A + \frac{\sqrt{1+m^2}}{4\pi}\left[2G\ln G\cdot\tan\theta\right]_{\theta_a}^{\theta_b} + \frac{\sqrt{1+m^2}}{2\pi}G\int_{\theta_a}^{\theta_b}\sec^2\theta\ln(\sec\theta)\, d\theta$$

$$= A + \frac{\sqrt{1+m^2}}{2\pi}(x_b-x_a)\ln G + \frac{\sqrt{1+m^2}}{2\pi}G\left[\tan\theta\ln(\sec\theta)\right]_{\theta_a}^{\theta_b} - \frac{\sqrt{1+m^2}}{2\pi}G\int_{\theta_a}^{\theta_b}\tan^2\theta\, d\theta$$

$\theta_a \Rightarrow \frac{x_a - g}{G}$

$\theta_b \Rightarrow \frac{x_b - g}{G}$

$$= A + B + \frac{\sqrt{1+m^2}}{4\pi}G\left[\tan\theta\ln(1+\tan^2\theta)\right]_{\theta_a}^{\theta_b} + \frac{\sqrt{1+m^2}}{2\pi}G\int_{\theta_a}^{\theta_b}(1-\sec^2\theta)\, d\theta$$

$$= A + B + \frac{\sqrt{1+m^2}}{4\pi}G\left[\frac{x_b-g}{G}\ln\left(1+\left(\frac{x_b-g}{G}\right)^2\right) - \frac{x_a-g}{G}\ln\left(1+\left(\frac{x_a-g}{G}\right)^2\right)\right]$$

$$+ \left[\frac{\sqrt{1+m^2}}{2\pi}(\theta_b-\theta_a) - \frac{\sqrt{1+m^2}}{2\pi}\frac{x_b-x_a}{G}\right]G$$

$$= A + B + C + D - E$$

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If $|m| = \infty$, $\displaystyle\int_{\Gamma_j} \frac{\ln r}{2\pi} ds_j = \frac{1}{2\pi}\int_{y_a}^{y_b} \ln(x_a^2 + y^2)\, dy$

$y = x_a \tan\theta$
$dy = x_a \sec^2\theta\, d\theta$

$$= \frac{1}{2\pi}\int_{\theta_a}^{\theta_b} \left[\ln(x_a^2) + \ln(\sec^2\theta)\right] \sec^2\theta\, d\theta$$

$$= \frac{\ln(x_a)^2}{2\pi}\,\frac{y_b - y_a}{x_a} + \frac{1}{2\pi}\int_{\theta_a}^{\theta_b} 2\sec^2\theta \ln\sec\theta\, d\theta$$

$$= A + \frac{1}{2\pi}\left[\tan\theta \ln(1+\tan^2\theta)\right]_{\theta_a}^{\theta_b} - \frac{1}{\pi}\int_{\theta_a}^{\theta_b} \tan^2\theta\, d\theta$$

$\theta_a \Rightarrow \frac{y_a}{x_a}$
$\theta_b \Rightarrow \frac{y_b}{x_a}$

$$= A + \frac{1}{2\pi}\left[\frac{y_b}{x_a}\ln\left(1+\left(\frac{y_b}{x_a}\right)^2\right) - \frac{y_a}{x_a}\ln\left(1+\left(\frac{y_a}{x_a}\right)^2\right)\right] + \frac{1}{\pi}\int_{\theta_a}^{\theta_b}(1-\sec^2\theta)\, d\theta$$

$$= A + B + \frac{1}{\pi}(\theta_b - \theta_a) - \frac{1}{\pi}\,\frac{y_b - y_a}{x_a}$$

$$\int_{\Gamma_j} \frac{dv_{ij}}{dn} ds_j = \int_{\Gamma_j} \frac{\cos\phi}{2\pi r} ds_j$$

$\cos\phi = \frac{\vec{x}\cdot\vec{n}_2}{|\vec{x}||\vec{n}_2|}$

$$= \frac{\sqrt{1+m^2}}{2\pi}\int_{x_a}^{x_b} \frac{\cos\phi}{\sqrt{x^2+y^2}} |dx|$$

$n_2 = -(y_b - y_a)\hat{i} + (x_b - x_a)\hat{j}$

$|n_2| = l$

$$= \frac{\sqrt{1+m^2}}{2\pi}\int_{x_a}^{x_b} \frac{-x(y_b-y_a) + y(x_b - x_a)}{(x^2+y^2)\sqrt{(x_b-x_a)^2 + (y_b-y_a)^2}} |dx|$$

$$= \frac{\sqrt{1+m^2}}{2\pi l}\int_{x_a}^{x_b} \frac{-x(y_b-y_a) + y(x_b-x_a)}{x^2+y^2} |dx|$$

$$= \frac{(x_b - x_a)\sqrt{1+m^2}}{4\pi l}\int_{-1}^{1} \frac{-x(y_b-y_a) + y(x_b-x_a)}{x^2+y^2} |dx'|$$

if $|m| = \infty$, $x_a = x_b$

$$\int_{\Gamma_j} \frac{dv_{ij}}{dn} ds_j = \frac{1}{2\pi}\int_{y_a}^{y_b} \frac{-x(y_b-y_a)|dy|}{(x^2+y^2)|y_b - y_a|} = \frac{y_b - y_a}{4\pi}\int_{-1}^{1} \frac{-x}{x^2+y^2} |dy'| \cdot \operatorname{sgn}(y_b - y_a)$$

Boundary Element Method

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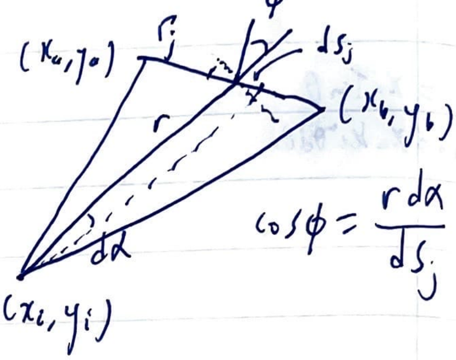
analytic solution:

$$\int_{\Gamma_j} \frac{\partial v_{ij}}{\partial n} ds_j$$

$$= \int_{\Gamma_j} \frac{\cos\phi}{2\pi r} ds_j$$

$$= \int_{\alpha_a}^{\alpha_b} \frac{d\alpha}{2\pi} = \frac{\alpha_b - \alpha_a}{2\pi}$$

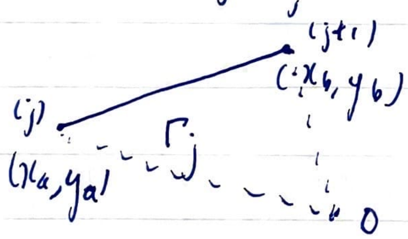


$$\cos\phi = \frac{r\,d\alpha}{ds_j}$$

Linear elements:

$$c_i u_i = -\oint_{\Gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds \quad , \quad \nabla^2 v = \delta$$

$$= -\sum_j \oint_{\Gamma_j}\left(v_{ij}\frac{\partial u}{\partial n} - u\frac{\partial v_{ij}}{\partial n}\right)ds_j$$



$$(r_a, r_b)$$

$$\int_{\Gamma_j} v_{ij}\frac{\partial u}{\partial n} ds_j = \frac{1}{2\pi}\int_{\Gamma_j} \ln(r)\left(\frac{\frac{\partial u_{j+1}}{\partial n} - \frac{\partial u_j}{\partial n}}{x_b - x_a}(x - x_a) + \frac{\partial u_j}{\partial n}\right) ds_j$$

$K = \sqrt{1+m^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$= \frac{K}{2\pi}\int_{x_a}^{x_b} \ln(r)\,\frac{x - x_a}{x_b - x_a}\,\frac{\partial u_{j+1}}{\partial n}|dx| + \frac{K}{2\pi}\int_{r_a}^{r_b} \ln(r)\,\frac{x_b - x}{x_b - x_a}\,\frac{\partial u_j}{\partial n}|dx|$$

$x' = \frac{2}{x_b - x_a}\left(x - \frac{x_a + x_b}{2}\right)$

$x = \frac{1}{2}\left((x_b - x_a)x' + x_a + x_b\right)$

$$= \frac{\partial u_{j+1}}{\partial n}\frac{K}{2\pi(x_b - x_a)}\int_{-1}^{1} \ln(r)\left[\frac{1}{2}\left((x_b - x_a)x' + x_a + x_b\right) - x_a\right]\frac{x_b - x_a}{2}|dx'|$$

$$+ \frac{\partial u_j}{\partial n}\frac{K}{2\pi(x_b - x_a)}\int_{-1}^{1} \ln(r)\left[x_b - \frac{1}{2}\left((x_b - x_a)x' + x_a + x_b\right)\right]\frac{x_b - x_a}{2}|dx'|$$

~~$x \to r'$, $y \to r$~~

~~if $r_a \to r_b$, $dr = -dr$~~

$$= \frac{\partial u_{j+1}}{\partial n}K\frac{x_b - x_a}{8\pi}\int_{-1}^{1} \ln(r)(x'+1)|dx'|$$

$$+ \frac{\partial u_j}{\partial n}K\frac{x_b - x_a}{8\pi}\int_{-1}^{1} \ln(r)(1 - x')|dx'|$$

Boundary Element Method

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$$\int_{\Gamma_j} u \frac{\partial v}{\partial n} ds_j = \frac{1}{2\pi}\int_{\Gamma_j} \frac{\cos\phi}{r}\left(\frac{u_{j+1}-u_j}{x_b-x_a}(x-x_a)+u_j\right)ds_j$$

$$= \frac{1}{2\pi}\int_{\Gamma_j} \frac{\vec{r}\cdot\hat{n}}{r^2}\left(\frac{u_{j+1}-u_j}{x_b-x_a}(x-x_a)+u_j\right)ds_j$$

$$= \frac{K}{2\pi}\int_{r_a}^{r_b} \frac{\vec{r}\cdot\hat{n}}{r^2}\left(\frac{x-x_a}{x_b-x_a}u_{j+1}+\frac{(x_b-x)}{x_b-x_a}u_j\right)|dx|$$

$$= \frac{K}{2\pi(x_b-x_a)}\int_{-1}^{1} \frac{\vec{r}\cdot\hat{n}}{r^2}\left[\left(\frac{x_b-x_a}{2}x'+\frac{x_b-x_a}{2}\right)u_{j+1} + \left(\frac{x_b-x_a}{2}-\frac{x_b-x_a}{2}x'\right)u_j\right]\frac{x_b-x_a}{2}|dx'|$$

$$= K\frac{x_b-x_a}{8\pi}\int_{-1}^{1}\frac{\vec{r}\cdot\hat{n}}{r^2}\left[(x'+1)u_{j+1}+(1-x')u_j\right]|dx'|$$

$$= u_{j+1}K\frac{x_b-x_a}{8\pi}\int_{-1}^{1}\frac{\vec{r}\cdot\hat{n}}{r^2}(x'+1)|dx'| + u_jK\frac{x_b-x_a}{8\pi}\int_{-1}^{1}\frac{\vec{r}\cdot\hat{n}}{r^2}(1-x')|dx'|$$

~~if $r_a > r_b$~~

~~$dr = dr$~~

~~u_y~~

$u = \frac{u_{j+1}-u_j}{L}t + u_j$

$x = \frac{x_b-x_a}{L}t + x_a$

$y = \frac{y_b-y_a}{L}t + y_a$

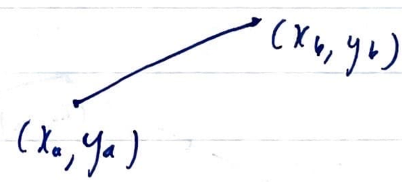
$t \in [0, L]$,

$\therefore u = (u_{j+1}-u_j)\frac{x-x_a}{x_b-x_a} + u_j$

---

special note:

$$ds = \sqrt{dx^2+dy^2}$$



$x = r\cos\theta \quad y = r\sin\theta$

$dx = \cos\theta dr - r\sin\theta d\theta$

$dy = \sin\theta dr + r\cos\theta d\theta$

$dx^2 + dy^2 = dr^2(\sin^2\theta+\cos^2\theta) + d\theta^2(r^2\sin^2\theta + r^2\cos^2\theta)$

$= dr^2 + r^2 d\theta^2$

$\therefore ds = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,dr$

$y = mx + c$, $\quad m = \tan\theta$

$r\sin\theta = mr\cos\theta$

$\sin\theta dr + r\cos\theta d\theta = m\cos\theta dr - mr\sin\theta d\theta$

$r(\cos\theta + m\sin\theta)\frac{d\theta}{dr} = m\cos\theta - \sin\theta = 0$

$\therefore r\frac{d\theta}{dr} = 0$

Boundary Element Method

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line segment

$$\therefore \int_{\Gamma_j} v_{ij}\, ds_j = \int_{r_a}^{r_b} \frac{\ln(r)}{2\pi}\, dr$$

$$= \frac{r_b - r_a}{4\pi} \int_{-1}^{1} \ln(r)\, dr'$$

$$\therefore \int_{\Gamma_j} \frac{\partial v_{ij}}{\partial n}\, ds_j = \int_{r_a}^{r_b} \frac{\cos\phi}{2\pi r}\, dr$$

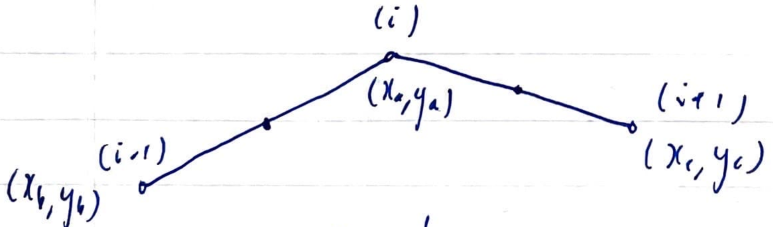
$$= \frac{r_b - r_a}{4\pi} \int_{-1}^{1} \frac{\vec{r}\cdot\hat{n}}{r^2}\, dr'$$

If $u = \text{const}$, $\frac{\partial u}{\partial n}$ must be zero over the closed boundary.

for unbounded boundary

$$\therefore H_{ii} = \hat{H}_{ii} + C_i = -\sum_{j \neq i} H_{ij}$$

$$\because \sum H = 0 \Leftarrow Hu = 0$$



$r_a = 0$
$r_b = l_{ab}$
$r_c = l_{ac}$

$$\cancel{G_{ii} = \frac{r_a - r_b}{8\pi} \int_{-1}^{1} \ln(r)(r'+1)\, dr' + \frac{r_c - r_a}{8\pi} \int_{-1}^{1} \ln(r)(1-r')\, dr'}$$

$$\cancel{= \frac{l_{ab}}{8\pi} \int_{-1}^{1} (r'+1) \ln\left[\frac{l_{ab} r' + l_{ab}}{2}\right] dr' + \frac{l_{ac}}{8\pi} \int_{-1}^{1} (1-r') \ln\left[\frac{l_{ac} r' + l_{ac}}{2}\right] dr'}$$

$$\cancel{= \frac{l_{ab}}{8\pi} \int_{-1}^{1} \left[(r'+1) \ln \frac{l_{ab}}{2} + (r'+1) \ln(r'+1)\right] dr' + \frac{l_{ac}}{8\pi} \int_{-1}^{1} \left[(1-r') \ln \frac{l_{ac}}{2} + (1-r') \ln(1-r')\right] dr'}$$

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$$G_{ii} = \frac{l_{ab}}{8\pi}\left[\cancel{\quad}\ \frac{r'^2}{2}+r'\right]_{-1}^{1}\ln\frac{l_{ab}}{2} + \frac{l_{ac}}{8\pi}\left[-\frac{r'^2}{2}+r'\right]_{-1}^{1}\ln\frac{l_{ac}}{2}$$

$$+\frac{l_{ab}}{8\pi}\int_0^2 r''\ln r''\,dr'' + \frac{l_{ac}}{8\pi}\int_0^2(2-r'')\ln r''\,dr''$$

X replaced by $\frac{l_{ab}}{8\pi}\ln\frac{l_{ab}}{2} + \frac{l_{ac}}{8\pi}\ln\frac{l_{ac}}{2}$

$$= \cancel{\frac{l_{ab}+l_{ac}}{4\pi}} + \frac{l_{ab}}{8\pi}\left[\frac{r''^2}{2}\ln r''\right]_0^2 - \frac{l_{ab}}{8\pi}\int_0^2\frac{r''}{2}dr'' + \frac{l_{ac}}{8\pi}\left[r''\ln r''\right]_0^2$$

$$-\frac{l_{ac}}{8\pi}\left[2r''\right]_0^2 + \cancel{\frac{l_{ac}}{8\pi}\left[\frac{2}{r''}\right]_0^2} - \frac{l_{ac}}{8\pi}\left[\frac{r''^2}{2}\ln r''\right]_0^2 + \frac{l_{ac}}{8\pi}\int_0^2\frac{r''}{2}dr''$$

$$\cancel{= \frac{l_{ab}+l_{ac}}{4\pi}}$$

$$= \frac{l_{ab}}{4\pi}\ln\frac{l_{ab}}{2} + \frac{l_{ac}}{4\pi}\ln\frac{l_{ac}}{2} + \frac{l_{ab}}{4\pi}\ln 2 - \frac{l_{ab}}{16\pi} + \frac{l_{ac}}{4\pi}\ln 2$$

$$-\frac{l_{ac}}{2\pi} - \frac{l_{ac}}{4\pi}\ln 2 + \frac{l_{ac}}{16\pi}$$

$=$

$$\# \ G_{ii} = \frac{1}{2\pi}\int_{r_b}^{r_a}\ln(r)\frac{r-r_b}{r_a-r_b}(-dr) + \frac{1}{2\pi}\int_{r_a}^{r_c}\ln(r)\frac{r_c-r}{r_c-r_a}dr$$

$$= \frac{1}{2\pi}\int_{l_{ab}}^{0}\ln(r)\frac{r-l_{ab}}{-l_{ab}}(-dr) + \frac{1}{2\pi}\int_0^{l_{ac}}\ln(r)\frac{l_{ac}-r}{l_{ac}}dr$$

$$= \frac{-1}{2\pi l_{ab}}\int_0^{l_{ab}}\ln(r)\cdot(r-l_{ab})\,dr + \frac{1}{2\pi l_{ac}}\int_0^{l_{ac}}\ln(r)\cdot(l_{ac}-r)\,dr$$

$$= \frac{-1}{2\pi l_{ab}}\left[\frac{r^2}{2}\ln(r) - \frac{r^2}{4} - l_{ab}r\ln(r) + l_{ab}r\right]_0^{l_{ab}}$$

$$+\frac{1}{2\pi l_{ac}}\left[l_{ac}r\ln(r) - l_{ac}r - \frac{r^2}{2}\ln(r) + \frac{r^2}{4}\right]_0^{l_{ac}}$$

$$= \frac{-1}{2\pi l_{ab}}\left(-\frac{1}{2}l_{ab}^2\ln(l_{ab}) + \frac{3}{4}l_{ab}^2\right) + \frac{1}{2\pi l_{ac}}\left(\frac{1}{2}l_{ac}^2\ln(l_{ac}) - \frac{3}{4}l_{ac}^2\right)$$

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$$\therefore G_{ii} = \frac{l_{ab}}{4\pi}\left(\ln(l_{ab}) - \frac{3}{2}\right) + \frac{l_{ac}}{4\pi}\left(\ln(l_{ac}) - \frac{3}{2}\right)$$

$$G_{i\,i+1} = \frac{1}{2\pi}\int_{r_a}^{r_c} \ln(r)\,\frac{r-r_a}{r_c-r_a}\,|dr|$$

$$= \frac{1}{2\pi l_{ac}}\int_0^{l_{ac}} \ln(r)\,r\,dr$$

$$= \frac{1}{2\pi l_{ac}}\left[\frac{r^2}{2}\ln(r) - \frac{r^2}{4}\right]_0^{l_{ac}}$$

$$= \frac{l_{ac}}{4\pi}\left(\ln(l_{ac}) - \frac{1}{2}\right)$$

$$G_{i\,i-1} = \frac{1}{2\pi}\int_{r_b}^{r_a} \ln(r)\,\frac{r_a-r}{r_a-r_b}\,(-dr)$$

$$= \frac{1}{2\pi l_{ab}}\int_0^{l_{ab}} \ln(r)\,r\,dr$$

$$= \frac{l_{ab}}{4\pi}\left[\ln(l_{ab}) - \frac{1}{2}\right]$$

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Boundary Element Method

$\nabla^2 u = f$

$u = u_0 + u_p$ ← particular solution

$\nabla^2 u_0 = 0$

$\nabla^2 F = f$

$$\therefore u = \int_\Omega v f \, dV - \int_\Gamma \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$

$$\int_\Omega \left(v \nabla^2 F - F \nabla^2 v \right) dV = \int_\Gamma \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds$$

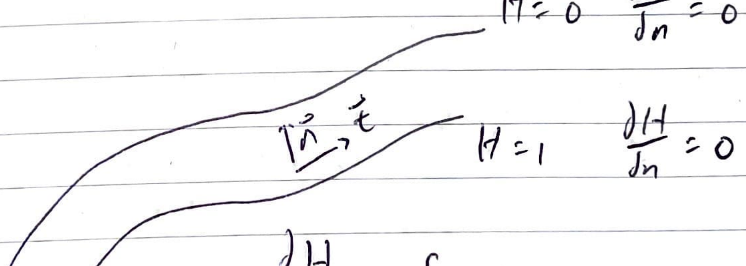
$$\int_\Omega v f \, dV = \int_\Omega F \delta \, dV + \int_\Gamma \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds$$

$$= F + \int_\Gamma \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds$$

if f is a divergence of a delta function, F is the Heaviside function

$\nabla H = \delta$

$\nabla^2 H = \nabla \cdot \delta = f$



$\frac{\partial H}{\partial n} = \delta$

$\frac{\partial^2 H}{\partial n^2} = \frac{\partial \delta}{\partial n}$

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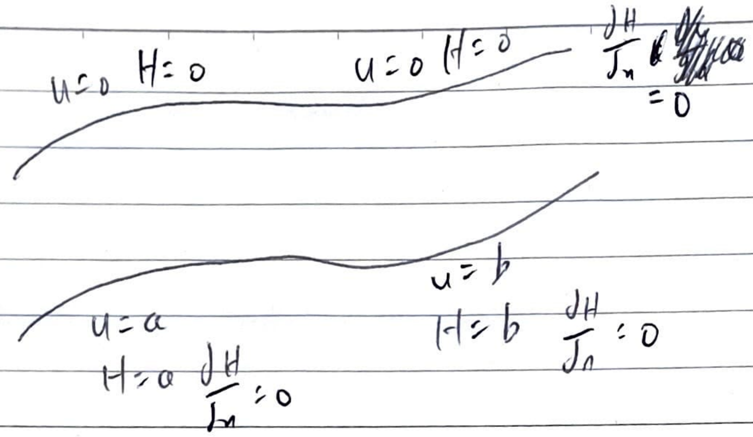
Boundary Element Method



$$u = F - \int_\Gamma \left[v\left(\frac{\partial u}{\partial n} - \frac{\partial F}{\partial n}\right) - (u-F)\frac{\partial v}{\partial n} \right] ds \quad , \quad F = H \text{ if } f = 0.5$$

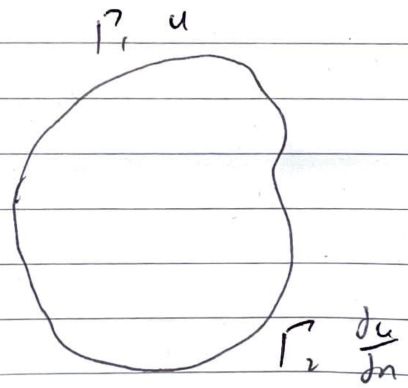
$$= F - \int_\Gamma v \frac{\partial u}{\partial n}\, ds$$

Dual Reciprocity Method

$\nabla^2 u = f$

$u = u_0 + u_p$ ← particular solution

↑ homogeneous solution



boundary condition:

$u_0 = u - u_p$ on Γ_1

$\frac{\partial u_0}{\partial n} = \frac{\partial u}{\partial n} - \frac{\partial u_p}{\partial n}$ on Γ_2

particular solution : $\nabla^2 u_p = f$

Boundary Element Method

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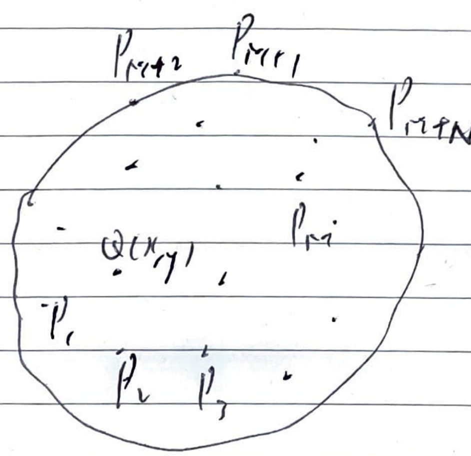
$$\therefore u = \int_{\Omega} v f \, dV - \int_{\Gamma} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$

evaluation of this domain integral



let $f(Q) = \sum_{j=1}^{M+N} a_j \phi_j(r_{jQ})$

$$r_{jQ} = \sqrt{(x_j - x)^2 + (y_j - y)^2}$$

ϕ is some radial basis function

$$\therefore f(P_i) = \sum_{j=1}^{M+N} a_j \phi_j(r_{ji})$$

$$f = \Phi a$$

$$\therefore \int_{\Omega_Q} v f \, dV_Q = \sum_j a_j \int_{\Omega_Q} v(P,Q) \phi_j(r_{jQ}) \, dV_Q$$

let $\nabla^2 w_j = \phi_j(r)$, w_j particular solution for $\phi_j(r)$

$$\int_{\Omega_Q} v(P,Q) \phi_j(r_{jQ}) \, dV_Q = \int_{\Omega_Q} v(P,Q) \nabla^2 w_j \, dV_Q$$

$$= \varepsilon(P) w_j(P) + \int_{\Gamma_q} \left[v(P,q) \frac{\partial w_j(q)}{\partial n} - w_j(q) \frac{\partial v(P,q)}{\partial q} \right] ds_q$$

$$\therefore \int_{\Omega_Q} v f \, dV_Q = \sum_j a_j \left\{ \varepsilon(P) w_j(P) + \int_{\Gamma} \left[v \frac{\partial w_j}{\partial n} - w_j \frac{\partial v}{\partial n} \right] ds_q \right\}$$

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Boundary Element Method

we can extend the domain of integral to the entire domain Ω, hence

$$\int_{\Omega} v f \, dV = \sum_{j=1}^{N+N'} a_j \left\{ \varepsilon(P) w_j(P) + \int_{\Gamma} \left[v(P,q) \frac{\partial w_j(q)}{\partial n} - w_j(q) \frac{\partial v(P,q)}{\partial n} \right] dS_q \right\}.$$